On a Special S-act Over Several Kinds of Semigroups

Leting Feng[1,a,*]

[1] College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, Gansu, China

a Email: fenglt0214@163.com

* Corresponding Author

Abstract: In this paper, we study some properties of a special S-acts $D(S) = (S \times S)_S$ over several kinds of semigroups. First, we establish relationships between the diagonal S-act $D(S)$ of the semigroup S and the flatness properties of the diagonal acts $D(M(S;\Lambda;P))$ of the Rees matrix semigroup $M(S;\Lambda;P)$. Then, we show that for semigroups S with local units by $(WF)'$ of diagonal S-acts. Finally, we ascertain the direct product $(S \times T,*)$ is a semigroup when $(S,\cdot)$ and $(T,\circ)$ are two semigroups with operation and give flatness properties of diagonal $(S \times T,*)$-act and we examine additional properties of the diagonal S-acts $D(S)$.

Keywords: the diagonal acts; semigroups with local units; Rees matrix semigroups; finite direct product semigroups.

1 INTRODUCTION

S-acts have been demonstrated to be an invaluable tool in the exploration of monoids from the perspective of external actions. Moreover, its theory has given birth to the non-additive homological algebra of monoids. Consequently, it is only logical to pose questions regarding the homological classification of monoids. In line with the theory of rings and modules, the set of outcomes that characterize monoids based on the properties of their associated S-acts is referred to as the homological classification of monoids. The diverse flatness properties, such as projectivity, flatness, weak flatness, principal weak flatness, and torsion freeness, have been extensively utilized in the homological classification of monoids (see [9,10]).

In the context where S represents a monoid, the diagonal act over S is invariably defined as the Cartesian product $S \times S$ with the right S-action $(s,t)u = (su,tu)$ for $s,t,u \in S$. Hence, this act shall be denoted as $D(S)$. A substantial amount of research has been conducted over the past decade, focusing on the inquiry of when diagonal acts are cyclic or finitely generated (see [3,4,6,7,18]). The flatness properties of diagonal acts over monoids was made by Bulman-Fleming and Gilmour in 2009, they provided, whenever possible, conditions on a monoid that characterize when its diagonal act exhibits a particular flatness property and ascertained the degree to which these generally distinct flatness properties can be differentiated from one another through examples of diagonal acts (see [2]).

In [14], authors persevere in the study of projective acts and explore the homological classification of semigroups with local units through the projectivity of their cyclic acts. Of greater significance is that, in the process of characterizing the intrinsic properties of semigroups with local units through the flatness attributes (namely, flatness, weak flatness, and principal weak flatness) of their associated acts, the employment of natural isomorphisms becomes essential. More detail homological classification for semigroups S with local units by flatness properties of S-acts please refer to [14].

In the domain of ring theory, it is a widely recognized fact that Morita equivalence sustains many significant properties (see [1]). In fact, Lawson in 2011 has demonstrated that several crucial subclasses of regular semigroups exhibit Morita invariance, and provided that these semigroups with local units (see [11,13]). The invariance of the majority of the properties under consideration has been proven under the circumstance where some type of local units exist within the semigroup. In several cases, a condition of this sort cannot be done away with. In [11], the study of Morita invariants in semigroups having some kinds of local units has been initiated, they proved that if S and T are two strongly Morita equivalent semigroups with weak local units then there is an isomorphism between their lattices of ideals which takes finitely generated ideals to finitely generated ideals and principal ideals to principal ideals.

The structure of completely simple semigroups has been expounded by D. Rees with reference to their maximal subgroups. Precisely, a semigroup is regarded as completely simple if and only if it is isomorphic to a Rees matrix semigroup predicated over a group. Nagy and Toth [16] in 2024 examined Rees matrix semigroups $M(S;\Lambda;P)$ over semigroups S, where Λ is an arbitrary non-empty set and P is an arbitrary mapping of Λ into S. They provided some theorems on connection between properties of semigroups and Rees matrix semigroups $M(S;\Lambda;P)$ over semigroups S.

The outline of this paper is as follows. In Sect. 3, we study for semigroups S with local units by flatness properties of diagonal S-acts. In Sect. 4, we discuss the connection between the diagonal S-act $D(S)$ of the semigroup S and the flatness properties of the diagonal acts $D(M(S;\Lambda;P))$ of the Rees matrix semigroup $M(S;\Lambda;P)$. In Sect. 5, we ascertain the direct product $(S\times T,*)$ is a semigroup when $(S,\cdot)$ and $(T,\circ)$ are two semigroups with operation and give flatness properties of diagonal $(S\times T,*)$-act.

2 PRELIMINARIES

We firstly review the concept of unitary. A right S-act A_S is called unitary if $AS=A$. The term ``Morita context" was adopted in the semigroup case by Talwar in [20]. A unitary Morita context is a six-tuple $(S,T,{}_SP_T,{}_TQ_S,\theta,\phi)$, where S and T are semigroups, ${}_SP_T\in_S Act_T$ and ${}_TQ_S\in_T Act_S$ are unitary biacts, and

$$\theta:_S(P\otimes_T Q)_S\to_S S_S\,,\quad \phi:_T(Q\otimes_S P)_T\to_T T_T$$

are biact morphisms such that, for every $p,p'\in P,\quad q,q'\in Q,$

$$\theta(p \otimes q)p' = p\phi(q \otimes p'), \quad q\theta(p \otimes q') = \phi(q \otimes p)q'.$$

Semigroups S and T are called strongly Morita equivalent, if there exists a unitary Morita context $(S, T, {}_S P_T, {}_T Q_S, \theta, \phi)$ such that the mappping θ and ϕ are surjective.

Next, we will introduce the types of local units that are required.

DEFINITION 2.1 ([19]) A semigroup S is said to have local units if, for any $s \in S$, there exist idempotents (not necessarily unique) $e, f \in E(S)$ such that $es = s = sf$.

DEFINITION 2.2 ([11]) A semigroup S is said to have common weak local units if, for any $s, t \in S$ there exists $u \in S$ such that $s = su$ and $t = tu$.

In [14], the concept of S-acts over semigroups is defined. Recall that a nonempty set A is called a right S-act (or right act over semigroup S), if there is a mapping from $A \times S$ to A, which maps (a, s) to as, such that $a(st) = (as)t$ for all $a \in A$, $s, t \in S$. We denoted this right S-act by A_S. The collection of all right S-acts, together with the S-homomorphisms, forms the category of right S-acts, which we denote by $Act - S$.

DEFINITION 2.3 ([15]) For any semigroup S, a right S-act A_S is called firm if the mapping $\mu_A: A \otimes_S S \longrightarrow A$, $a \otimes s \mapsto as$ is bijective. A semigroup S is called firm if it is firm as a right S-act. The category of all firm right S-acts is denoted by $FAct - S$.

In the category $FAct - S$, flatness, weak flatness and principal weak flatness are formulated and readers can refer to [\cite{L-K-Z-2021}] for more details and related characterizations of these definitions.

LEMMA 2.4 ([14]) Let $A \in UAct - S$ and $B \in S - act$. Then $a \otimes b = a' \otimes b'$ for any $a, a' \in A_S$, $b, b' \in_S B$ if and only if there exist $a_1, \cdots, a_n \in A$, $b_2, \cdots, b_n \in B$, $s_1, t_1, \cdots, s_n, t_n \in S$ such that

$$\begin{array}{cc} & a = a_1 s_1 \\ a_1 t_1 = a_2 s_2 & s_1 b = t_1 b_2 \\ a_2 t_2 = a_3 s_3 & s_2 b_2 = t_2 b_3 \\ \vdots & \vdots \\ a_n t_n = a' & s_n b_n = t_n b'. \end{array}$$

And we will use the next conditions for an S-act A_S over semigroups which appeared in [12].

A right S-act A_S is said to satisfy Condition (P), if for any $a, a' \in A_S$, $s, s' \in S, as = a's'$, then implies that there exist $a'' \in A_S$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs'$.

A right S-act A_S is said to satisfy Condition (E), if for any $a \in A_S$, $s, s' \in S$, $as = as'$, then implies that there exist $a' \in A_S$, $u \in S$ such that $a = a'u$, $us = us'$.

3 THE DIAGONAL S-ACTS OVER REES MATRIX SEMIGROUPS

In this subsection, we will attempt to discuss the connection between the diagonal S-act $D(S)$ of the semigroup S and the flatness properties of the diagonal acts $D(M(S; \Lambda; P))$ of the Rees matrix semigroup $M(S;\Lambda;P)$.

THEOREM 3.1 Let $R = M(S;\Lambda;P)$ be a Rees matrix semigroup over a semigroup S. Then the diagonal R-act $D(R)$ satisfies Condition (P) if and only if the diagonal S-act $D(S)$ satisfies Condition (P).

PROOF Assume that $(a,b)P(\lambda)s = (a',b')P(\lambda)s' \in D(S)$ for any $a,a',b,b',s,s' \in S$, $\lambda \in \Lambda$. Then $aP(\lambda)s = a'P(\lambda)s'$, $bP(\lambda)s = b'P(\lambda)s'$, imply that

$$((a,\lambda),(b,\lambda))(s,\xi) = ((a',\lambda),(b',\lambda))(s',\xi) \in D(R).$$

Since the diagonal R-act satisfies Condition (P), there exist $((a'',\lambda),(b'',\lambda)) \in D(R)$, $(u,\lambda),(v,\lambda) \in R$, such that

$$((a,\lambda),(b,\lambda)) = ((a'',\lambda),(b'',\lambda))(u,\lambda),$$
$$((a',\lambda),(b',\lambda)) = ((a'',\lambda),(b'',\lambda))(v,\lambda),$$

And $(u,\lambda)(s,\xi) = (v,\lambda)(s',\xi)$. And so

$$(a,b) = (a'',b'')P(\lambda)u,$$
$$(a',b') = (a'',b'')P(\lambda)v,$$

And $P(\lambda)us = P(\lambda)vs'$. Therefore, the diagonal S-act $D(S)$ satisfies Condition (P).

Conversely, assume that $((a,\lambda),(b,\lambda))(s,\xi) = ((a',\lambda),(b',\lambda))(s',\xi) \in D(R)$. Then $aP(\lambda)s = a'P(\lambda)s'$, $bP(\lambda)s = b'P(\lambda)s'$. We get

$$(a,b)P(\lambda)s = (a',b')P(\lambda)s' \in D(S).$$

Since the diagonal S-act $D(S)$ satisfies Condition (P), there exist $(a''P(\lambda),b''P(\lambda)) \in D(S)$, $u,v \in S$, such that

$$(a,b) = (a''P(\lambda),b''P(\lambda))u,$$
$$(a',b') = (a''P(\lambda),b''P(\lambda))v,$$

And $uP(\lambda)s = vP(\lambda)s'$, and hence

$$((a,\lambda),(b,\lambda)) = ((a'',\lambda),(b'',\lambda))(u,\lambda),$$
$$((a',\lambda),(b',\lambda)) = ((a'',\lambda),(b'',\lambda))(v,\lambda),$$

And $(u,\lambda)(s,\xi) = (v,\lambda)(s',\xi)$, Thus, the diagonal R-act D(R) satisfies Condition (P).

THEOREM 3.2 Let $R = M(S;\Lambda;P)$ be a Rees matrix semigroup over a semigroup S. Then the diagonal R-act $D(R)$ satisfies Condition (E) if and only if the diagonal S-act $D(S)$ satisfies Condition (E).

PROOF Assume that $(a,b)P(\lambda)s = (a,b)P(\lambda)s' \in D(S)$ for any $a,b,s,s' \in S$, $\lambda \in \Lambda$. Then $aP(\lambda)s = aP(\lambda)s'$, $bP(\lambda)s = bP(\lambda)s'$, it follows that

$$((a,\lambda),(b,\lambda))(s,\xi) = ((a,\lambda),(b,\lambda))(s',\xi) \in D(R).$$

Since the diagonal R-act $D(R)$ satisfies Condition (E), there exist $((a',\lambda),(b',\lambda)) \in D(R)$, $(u,\lambda) \in R$, such that

$$((a,\lambda),(b,\lambda)) = ((a',\lambda),(b',\lambda))(u,\lambda),$$

And $(u,\lambda)(s,\xi) = (u,\lambda)(s',\xi)$. So we get

$$(a,b) = (a',b')P(\lambda)u,$$

And $P(\lambda)us = P(\lambda)us'$. Thus, the diagonal S-act $D(S)$ satisfies Condition (E).

Conversely, assume that $((a,\lambda),(b,\lambda))(s,\xi) = ((a,\lambda),(b,\lambda))(s',\xi) \in D(R)$ for any $a,b,s,s' \in S$, $\lambda,\xi \in \Lambda$. Then $aP(\lambda)s = aP(\lambda)s'$, $bP(\lambda)s = bP(\lambda)s'$. It follows that

$$(a,b)P(\lambda)s = (a',b')P(\lambda)s' \in D(S).$$

Since the diagonal S-act $D(S)$ satisfies Condition (E), there exist $(a'P(\lambda),b'P(\lambda)) \in D(S)$, $u \in S$, such that

$$(a,b) = (a'P(\lambda),b'P(\lambda))u,$$

And $uP(\lambda)s = uP(\lambda)s'$. And hence

$$((a,\lambda),(b,\lambda)) = ((a',\lambda),(b',\lambda))(u,\lambda),$$

And $(u,\lambda)(s,\xi) = (u,\lambda)(s',\xi)$, Therefore, the diagonal R-act $D(R)$ satisfies Condition (E).

DEFINITION 3.3 A right S-act A_S is said to satisfy Condition (PWP), if for any $a,a' \in A_S$, $s \in S$, $as = a's$, then implies that there exist $a'' \in A_S$, $u,v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs$.

THEOREM 3.4 Let $R = M(S;\Lambda;P)$ be a Rees matrix semigroup over a semigroup S. Then the diagonal R-act $D(R)$ satisfies Condition (PWP) if and only if the diagonal S-act $D(S)$ satisfies Condition (PWP).

PROOF Assume that $(a,b)P(\lambda)s = (a',b')P(\lambda)s \in D(S)$ for any$a,a',b,b',s \in S$, $\lambda \in \Lambda$. Then $aP(\lambda)s = a'P(\lambda)s$, $bP(\lambda)s = b'P(\lambda)s$, imply that

$$((a,\lambda),(b,\lambda))(s,\xi) = ((a',\lambda),(b',\lambda))(s,\xi) \in D(R).$$

Since the diagonal R-act $D(R)$ satisfies Condition (PWP), there exist $((a'',\lambda),(b'',\lambda)) \in D(R)$, $(u,\lambda),(v,\lambda) \in R$, such that

$$((a,\lambda),(b,\lambda)) = ((a'',\lambda),(b'',\lambda))(u,\lambda),$$

$$((a',\lambda),(b',\lambda)) = ((a'',\lambda),(b'',\lambda))(v,\lambda),$$

And $(u,\lambda)(s,\xi) = (v,\lambda)(s,\xi)$. And so

$$(a,b) = (a'',b'')P(\lambda)u,$$

$$(a',b') = (a'',b'')P(\lambda)v,$$

And $P(\lambda)us = P(\lambda)vs'$. Thus, the diagonal S-act $D(S)$ satisfies Condition (PWP).

Conversely, assume that $((a,\lambda),(b,\lambda))(s,\xi) = ((a',\lambda),(b',\lambda))(s,\xi) \in D(R)$. Then $aP(\lambda)s = a'P(\lambda)s$, $bP(\lambda)s = b'P(\lambda)s$. It follows that

$$(a,b)P(\lambda)s=(a',b')P(\lambda)s\in D(S).$$

Since the diagonal S-act $D(S)$ satisfies Condition (PWP), there exist $(a''P(\lambda),b''P(\lambda))\in D(S)$, $u,v\in S$, such that

$$(a,b)=(a''P(\lambda),b''P(\lambda))u,$$
$$(a',b')=(a''P(\lambda),b''P(\lambda))v,$$

And $uP(\lambda)s=vP(\lambda)s$. We get

$$((a,\lambda),(b,\lambda))=((a'',\lambda),(b'',\lambda))(u,\lambda),$$
$$((a',\lambda),(b',\lambda))=((a'',\lambda),(b'',\lambda))(v,\lambda),$$

And $(u,\lambda)(s,\xi)=(v,\lambda)(s,\xi)$, Thus, the diagonal R-act $D(R)$ satisfies Condition (PWP).

THEOREM 3.5 Let $R=M(S;\Lambda;P)$ be a Rees matrix semigroup over a semigroup S. Then the diagonal R-act $D(R)$ is principally weakly flat if and only if the diagonal S-act $D(S)$ is also principally weakly flat.

PROOF Assume that $(a,b)P(\lambda)s=(a',b')P(\lambda)s$ for any $a,a',b,b',s\in S$, $\lambda\in\Lambda$. Then $aP(\lambda)s=a'P(\lambda)s$, $bP(\lambda)s=b'P(\lambda)s$, it follows that

$$((a,\lambda),(b,\lambda))(s,\xi)=((a',\lambda),(b',\lambda))(s,\xi)\in D(R).$$

Since the diagonal R-act $D(R)$ is principally weakly flat, implies $((a,\lambda),(b,\lambda))\otimes(s,\xi)=((a',\lambda),(b',\lambda))\otimes(s,\xi)\in D(R)\otimes R(s,\xi)$, by [14, Lemma 2.1], there exist $((a_1,\lambda),(b_1,\lambda)),\cdots,((a_{i-1},\lambda),(b_{i-1},\lambda))\in D(R)$, $(u_1,\lambda),\cdots,(u_i,\lambda),(s_1,\xi_1),\cdots,(s_i,\xi_i)\in R$, such that

$$\begin{array}{ll} & (u_1,\lambda)(s,\xi)=(s,\xi)\\ ((a,\lambda),(b,\lambda))(u_1,\lambda)=((a_1,\lambda),(b_1,\lambda))(s_1,\xi_1) & (u_2,\lambda)(s,\xi)=(s_1,\xi_1)(s,\xi)\\ ((a_1,\lambda),(b_1,\lambda))(u_2,\lambda)=((a_2,\lambda),(b_2,\lambda))(s_2,\xi_2) & (u_3,\lambda)(s,\xi)=(s_2,\xi_2)(s,\xi)\\ \vdots & \vdots\\ ((a_{i-1},\lambda),(b_{i-1},\lambda))(u_{i-1},\lambda)=((a',\lambda),(b',\lambda))(s_i,\xi_i) & (s,\xi)=(s_i,\xi_i)(s,\xi)\end{array}$$

So we obtain

$$\begin{array}{ll} & u_1P(\lambda)s=s\\ (a,b)P(\lambda)u_1=(a_1,b_1)P(\lambda)s_1 & u_2P(\lambda)s=s_1P(\xi_1)s\\ (a_1,b_1)P(\lambda)u_2=(a_2,b_2)P(\lambda)s_2 & u_3P(\lambda)s=s_2P(\xi_2)s\\ \vdots & \vdots\\ (a_{i-1},b_{i-1})P(\lambda)u_{i-1}=(a',b')P(\lambda)s_i & s=s_iP(\xi_i)s\end{array}$$

Thus, $(a,b)\otimes P(\lambda)s=(a',b')\otimes P(\lambda)s$ holds in $D(S)\otimes Ss$. Therefore, the diagonal S-act $D(S)$ is also principally weakly flat.

Conversely, assume that

$$((a,\lambda),(b,\lambda))(s,\lambda)=((a',\lambda),(b',\lambda))(s,\lambda)\in D(R).$$

for any $a, a', b, b', s \in S$, $\lambda, \xi \in \Lambda$. Then $aP(\lambda)s = a'P(\lambda)s$, $bP(\lambda)s = b'P(\lambda)s$, it follows that

$$(a, b)P(\lambda)s = (a', b')P(\lambda)s \in D(S).$$

Since the diagonal S-act $D(S)$ is also principally weakly flat, implies $(a, b) \otimes P(\lambda)s = (a', b') \otimes P(\lambda)s \in D(S) \otimes Ss$, by [14, Lemma 2.1], there exist $(a_1, b_1), \cdots, (a_{i-1}, b_{i-1}) \in D(S)$, $y_1, \cdots, y_i, s_1, \cdots, s_i) \in S$, such that

$$\begin{array}{ll} & y_1P(\lambda)s = s \\ (a, b)P(\lambda)y_1 = (a_1, b_1)P(\lambda)s_1 & y_2P(\lambda)s = s_1P(\lambda)s \\ (a_1, b_1)P(\lambda)y_2 = (a_2, b_2)P(\lambda)s_2 & y_3P(\lambda)s = s_2P(\lambda)s \\ \vdots & \vdots \\ (a_{i-1}, b_{i-1})P(\lambda)y_{i-1} = (a', b')P(\lambda)s_i & s = s_iP(\lambda)s \end{array}$$

We get

$$\begin{array}{ll} (y_1, \lambda)(s, \lambda) = (s, \lambda) & \\ ((a, \lambda), (b, \lambda))(y_1, \lambda) = ((a_1, \lambda), (b_1, \lambda))(s_1, \lambda) & (y_2, \lambda)(s, \lambda) = (s_1, \lambda)(s, \lambda) \\ ((a_1, \lambda), (b_1, \lambda))(y_2, \lambda) = ((a_2, \lambda), (b_2, \lambda))(s_2, \lambda) & (y_3, \lambda)(s, \lambda) = (s_2, \lambda)(s, \lambda) \\ \vdots & \vdots \\ ((a_{i-1}, \lambda), (b_{i-1}, \lambda))(y_{i-1}, \lambda) = ((a', \lambda), (b', \lambda))(s_i, \lambda) & (s, \lambda) = (s_i, \lambda)(s, \lambda) \end{array}$$

Thus, $((a, \lambda), (b, \lambda)) \otimes (s, \lambda) = ((a', \lambda), (b', \lambda)) \otimes (s, \lambda)$ holds in $D(R) \otimes R(s, \lambda)$. Therefore, the diagonal R-act $D(R)$ is principally weakly flat.

4 THE DIAGONAL ACTS OVER SEMIGROUPS WITH LOCAL UNITS

In Morita theory, the research on Morita invariants is extremely crucial, as it reflects the special relationship between semigroups S and T. During the study of diagonal acts, we have obtained an interesting conclusion, that is, the cyclicity invariants of the diagonal acts of strongly Morita equivalent semigroups with common joint weak local units. Next, we will present a simple and direct verification.

PROPOSITION 4.1 Let S and T be semigroups with common joint weak local units. If S and T are strongly Morita equivalent and the diagonal S-act $D(S)$ is cyclic, then the diagonal T-act $D(T)$ is also cyclic.

PROOF Take $t_1, t_2 \in T$. There exist $u, v \in T$ such that $t_1 = ut_1v$, and $t_2 = ut_2v$. Let $q_u, q_v \in Q$, $p_u, p_v \in P$, such that $u = \phi(q_u \otimes p_u)$, $v = \phi(q_v \otimes p_v)$. Since $\theta(p_u \otimes t_1q_v), \theta(p_u \otimes t_2q_v) \in S$, $D(S)$ is cyclic, there exists $\theta(p_u \otimes tq_v), \theta(p_u \otimes t'q_v) \in S$, such that $(\theta(p_u \otimes t_1q_v), \theta(p_u \otimes t_2q_v)) = (\theta(p_u \otimes tq_v), \theta(p_u \otimes t'q_v))S$. Putting $s = \theta(p \otimes q)$. The rest is straightforward checking:

$$(t_1, t_2) = (ut_1v, ut_2v) = (\phi(q_u \otimes p_u)t_1\phi(q_v \otimes p_v),\ \phi(q_u \otimes p_u)t_2\phi(q_v \otimes p_v))$$

$$= (\phi(q_u \otimes p_u\phi(t_1q_v \otimes p_v)),\ \phi(q_u \otimes p_u\phi(t_2q_v \otimes p_v)))$$
$$= (\phi(q_u \otimes \theta(p_u \otimes t_1q_v)p_v),\ \phi(q_u \otimes \theta(p_u \otimes t_2q_v)p_v))$$
$$= (\phi(q_u \otimes \theta(p_u \otimes tq_v)sp_v),\ \phi(q_u \otimes \theta(p_u \otimes t'q_v)sp_v))$$
$$= (\phi(q_u \otimes \theta(p_u \otimes tq_v)\theta(p \otimes q)p_v),\ \phi(q_u \otimes \theta(p_u \otimes t'q_v)\theta(p \otimes q)p_v))$$
$$= (\phi(q_u \otimes \theta(p_u \otimes tq_v)p\phi(q \otimes p_v)),\ \phi(q_u \otimes \theta(p_u \otimes t'q_v)p\phi(q \otimes p_v)))$$
$$= (\phi(q_u \otimes p_u\phi(tq_v \otimes p)\phi(q \otimes p_v)),\ \phi(q_u \otimes p_u\phi(t'q_v \otimes p)\phi(q \otimes p_v)))$$
$$= (\phi(q_u \otimes p_u)t\phi(q_v \otimes p)\phi(q \otimes p_v),\ \phi(q_u \otimes p_u)t'\phi(q_v \otimes p)\phi(q \otimes p_v))$$
$$= (ut\phi(q_v \otimes p)\phi(q \otimes p_v),\ ut'\phi(q_v \otimes p)\phi(q \otimes p_v))$$
$$= (ut, ut')(\phi(q_v \otimes p)\phi(q \otimes p_v)) \in (ut, ut')S$$

So we get $T \times T = (ut, ut')T$, that is, $D(T)$ is cyclic.

DEFINITION 4.2 A left S-act ${}_SA \in FAct - S$ is called locally cyclic, if for any $a_1, a_2 \in_S A$ there exists $z \in_S A$ such that $a_1, a_2 \in Sz$. A locally cyclic left ideal of S is called locally principal.

For any semigroup S, an S-act A_S is called $(WF)'$, if the functor $A_S \otimes_S -$ preserves all monomorphisms from left ideals of S of the form $J = Ss \cup St,\ sz = tz(s, t, z \in S)$ into S.

According to [8], it is mentioned that $A_S \in FAct - S$ is $(WF)'$ if $as = a't$ and $sz = tz$ for $a, a' \in A_S$ and $s, t, z \in S$ imply $a \otimes s = a' \otimes t$ in the tensor product $A_S \otimes_S (Ss \cup St)$. Then we see every $(WF)'$ right S-act is principally weakly flat.

PROPOSITION 4.3 For any semigroup S and $A \in FAct - S$. Then A_S is $(WF)'$, if and only if it is principally weakly flat and satisfies Condition $(W_{(WF)'})$: If $as = a't, sz = tz$ for $a, a' \in A_S$ and $s, t, z \in S$, then there exist $a'' \in A_S,\ w \in Ss \cap St$ such that $as = a't = a''w$.

PROOF Suppose that A_S is $(WF)'$, and let $as = a't,\ s, t, z \in S$. By assumption, $a \otimes s = a' \otimes t$ in the tensor product $A_S \otimes_S (Ss \cup St)$. Thus there exist $s_1, \cdots, s_k, t_1, \cdots, t_k \in S,\ u_1, \cdots, u_k \in Ss \cup St,\ a_1, \cdots, a_{k-1} \in A_S$

$$s_1u_1 = s$$
$$as_1 = a_1t_1 \qquad s_2u_2 = t_1u_1$$
$$a_1s_2 = a_2t_2 \qquad s_3u_3 = t_2u_2$$
$$\vdots \qquad\qquad \vdots$$
$$a_{k-1}s_k = a't_k \qquad t = t_ku_k.$$

Let i be the first index such that $u_i \in St$. If i=1, then $s = s_1u_1 \in St$, and so $s = vt$, for some $v \in S$, thus we can take $w = s$ and $a'' = a$. Suppose now that i>1. Then $u_{i-1} \in Ss$, and since by the above tossing $s_iu_i = t_{i-1}u_{i-1}$, then $w = s_iu_i \in Ss \cap St$ and so $as = as_1u_1 = a_1t_1u_1 = \cdots = a_{i-1}s_iu_i = a_{i-1}w$, thus Condition $(W_{(WF)'})$ holds for $a'' = a_{i-1}$. It is obvious that A_S is principally weakly flat because A_S is $(WF)'$.

Conversely. Let $as = a't,\ sz = tz$ for $a, a' \in A_S,\ s, t, z \in S$. Since A_S satisfies Condition $(W_{(WF)'})$, there exist $a'' \in A_S,\ u, v \in S$ such that $as = a''(us),\ a't = a''(vt)$ and $us = vt$. Since A_S is principally weakly flat we have $a \otimes s = a''u \otimes s \in A_S \otimes_S Ss$, and $a' \otimes t = a''v \otimes t \in A_S \otimes_S St$. Hence, $a \otimes s = a''u \otimes s = a'' \otimes us = a'' \otimes vt = a''v \otimes t = a' \otimes t$ in $A_S \otimes_S (Ss \cup St)$.

DEFINITION 4.4 A semigroup S is said to finitely $(WF)'$ coherent semigroup, If finite products of acts which satisfying Condition $(WF)'$ satisfy Condition $(WF)'$.

LEMMA 4.5 The diagonal S-act $D(S)$ satisfies Condition $(WF)'$ if and only if $D(S)$ is principally weakly flat, and for any $s, t, z \in S,\ sz = tz$, either the left ideal $Ss \cap St$ of S is non-empty or is locally principal.

THEOREM 4.6 A semigroup S is finitely $(WF)'$ coherent if and only if the diagonal S-act $D(S)$ $(S^n, n > 1)$ satisfies Condition $(WF)'$.

PROOF Suppose that the diagonal S-act $D(S)$ satisfies Condition $(WF)'$, and we consider the following two S-acts A_S and B_S which satisfying Condition $(WF)'$. Since $D(S)$ is principally weakly flat, $(A_S \times B_S)$,too. Thus, we only need to verify $(A_S \times B_S)$ satisfies Condition $(W_{(WF)'})$. Let $(a, b)s = (a', b')t$ and $sz = tz$ for any $a, a' \in A_S,\ b, b' \in B_S,\ s, t, z \in S$, from A_S and B_S which satisfying Condition $(WF)'$, there exist $a'' \in A_S, b'' \in B_S$ and $u, v \in Ss \cap St,\ x, y \in S$ such that $u = xw$, $v = yw$. Therefore, $(a, b)s = (a', b')t = (a''u, b''v) = (a''x, b''y)$. So, $(A_S \times B_S)$ satisfies Condition $(W_{(WF)'})$. It follows from that $(A_S \times B_S)$ satisfies Condition $(WF)'$, and draw a conclusion according to an inductive process.

Thus, we naturally obtain the following Corollary.

COROLLARY 4.7 Let S be a semigroup. The diagonal S-act $D(S)$ satisfies Condition $(WF)'$ if and only if for any $a, b, s \in S,\ as = bs$, implies $(1, a) \otimes s = (1, b) \otimes s \in D(S) \otimes Ss$ and for $x, y, z \in S,\ xz = yz,\ Sx \cap Sy$ if nonempty, is locally principal.

5 THE DIAGONAL ACTS OVER FINITE DIRECT PRODUCT SEMIGROUPS

Now let $(S, \cdot)$ and $(T, \circ)$ be two semigroups, for any $s_1, s_2 \in S,\ t_1, t_2 \in T$, direct product $(S \times T, *)$ with operation

$$(s_1, t_1) * (s_2, t_2) = (s_1 \cdot s_2, t_1 \circ t_2),$$

Then $(S \times T, *)$ is also a semigroup.

PROPOSITION 5.1 Let $(S, \cdot)$ and $(T, \circ)$ be two semigroups. Then the diagonal $(S \times T)$-act $D(S \times T)$ is principally weakly flat if and only if both the diagonal S-act $D(S)$ and the diagonal T-act $D(T)$ are principally weakly flat.

PROOF Suppose that the diagonal $(S \times T)$-act $D(S \times T)$ is principally weakly flat, we claim that both the diagonal S-act $D(S)$ and the diagonal T-act $D(T)$ are principally weakly flat. In fact, we firstly prove that the diagonal S-act $D(S)$ is principally weakly flat. The case of the diagonal T-act $D(T)$ is similar. Let $(s, \bar{s})u = (s', \bar{s'})u \in D(S), u \in S$, and $(t, t) \in D(T)$, $v \in T$. Then

$$\big((s, t), (\bar{s}, t)\big) * (u, v) = \big((s', t), (\bar{s'}, t)\big) * (u, v) \in D(S \times T).$$

By assumption, the diagonal $(S \times T)$-act $D(S \times T)$ is principally weakly flat, from [14, Lemma 2.1], there exist $\big((s_i, t_i), (\bar{s}_i, \bar{t}_i)\big) \in D(S \times T), (p_i, q_i), (x_i, y_i) \in S \times T, i = 1,2, \cdots, k$ such that

$$\begin{array}{ll} & (p_1, q_1) * (u, v) = (u, v) \\ \big((s, t), (\bar{s}, t)\big) * (p_1, q_1) = \big((s_1, t_1), (\bar{s}_1, \bar{t}_1)\big) * (x_1, y_1) & (p_2, q_2) * (u, v) = (x_1, y_1) * (u, v) \\ \big((s_1, t_1), (\bar{s}_1, \bar{t}_1)\big) * (p_2, q_2) = \big((s_2, t_2), (\bar{s}_2, \bar{t}_2)\big) * (x_2, y_2) & (p_3, q_3) * (u, v) = (x_2, y_2) * (u, v) \\ \vdots & \vdots \\ \big((s_{k-1}, t_{k-1}), (\bar{s}_{k-1}, \bar{t}_{k-1})\big) * (p_k, q_k) = \big((s', t), (\bar{s'}, t)\big) * (x_k, y_k) & (u, v) = (x_k, y_k) * (u, v) \end{array}$$

Then, we get

$$\begin{array}{ll} & p_1 \cdot u = u \\ (s, \bar{s}) \cdot p_1 = (s_1, \bar{s}_1) \cdot x_1 & p_2 \cdot u = x_1 \cdot u \\ (s_1, \bar{s}_1) \cdot p_2 = (s_2, \bar{s}_2) \cdot x_2 & p_3 \cdot u = x_2 \cdot u \\ \vdots & \vdots \\ (s_{k-1}, \bar{s}_{k-1}) \cdot p_k = (s', \bar{s'}) \cdot x_k & u = x_k \cdot u \end{array}$$

Thus, $(s, \bar{s}) \otimes u = (s', \bar{s'}) \otimes u \in D(S) \otimes Su$. Therefore, the diagonal S-act $D(S)$ is principally weakly flat.

Likewise, suppose that $(t, \bar{t}) \circ v = (t', \bar{t'}) \circ v \in D(T)$, $v \in T$ and $(s, s) \in D(S)$, $u \in S$. Then

$$\big((s, t), (s, \bar{t})\big) * (u, v) = \big((s, t'), (s, \bar{t'})\big) * (u, v) \in D(S \times T).$$

Thus, the diagonal T-act $D(T)$ is also principally weakly flat.

Conversely, suppose that

$$\big((s, t), (\bar{s}, \bar{t})\big) * (u, v) = \big((s', t'), (\bar{s'}, \bar{t'})\big) * (u, v) \in D(S \times T).$$

Then $(s,\bar{s})\cdot u=(s',\bar{s'})\cdot u\in D(S)$, $(t,\bar{t})\circ v=(t',\bar{t'})\circ v\in D(T)$. Since the diagonal S-act $D(S)$ and the diagonal T-act~ $D(T)$are principally weakly flat, from [14, Lemma 2.1], there exist $n\in N$and $p_i,x_i\in S$, $(s_i,\bar{s}_i)\in D(S)$, $u\in S, i=1,2,\cdots,n$ such that

$$\begin{array}{ll} & p_1\cdot u=u \\ (s,\bar{s})\cdot p_1=(s_1,\bar{s}_1)\cdot x_1 & p_2\cdot u=x_1\cdot u \\ (s_1,\bar{s}_1)\cdot p_2=(s_2,\bar{s}_2)\cdot x_2 & p_3\cdot u=x_2\cdot u \\ \vdots & \vdots \\ (s_{n-1},\bar{s}_{n-1})\cdot p_n=(s',\bar{s'})\cdot x_n & u=x_n\cdot u \end{array}$$

and there exist $m\in N$ and $q_i,y_i\in T,(t_i,\bar{t}_i)\in D(T),v\in T,i=1,2,\cdots,m$ such that

$$\begin{array}{ll} & q_1\circ v=v \\ (t,\bar{t})\circ q_1=(t_1,\bar{t}_1)\circ y_1 & q_2\circ v=y_1\circ v \\ (t_1,\bar{t}_1)\circ q_2=(t_2,\bar{t}_2)\circ y_2 & q_3\circ v=y_2\circ v \\ \vdots & \vdots \\ (t_{m-1},\bar{t}_{m-1})\circ q_m=(t',\bar{t'})\circ y_m & v=y_m\circ v \end{array}$$

So, we consider the following three cases:

Case1: If $n=m$, then have

$$\begin{array}{ll} & (p_1,q_1)*(u,v)=(u,v) \\ ((s,t),(\bar{s},\bar{t}))*(p_1,q_1)=((s_1,t_1),(\bar{s}_1,\bar{t}_1))*(x_1,y_1) & (p_2,q_2)*(u,v)=(x_1,y_1)(u,v) \\ ((s_1,t_1),(\bar{s}_1,\bar{t}_1))*(p_2,q_2)=((s_2,t_2),(\bar{s}_2,\bar{t}_2))*(x_2,y_2) & (p_3,q_3)*(u,v)=(x_2,y_2)*(u,v) \\ \vdots & \vdots \\ ((s_{n-1},t_{n-1}),(\bar{s}_{n-1},\bar{t}_{n-1}))*(p_n,q_n)=((s',t'),(\bar{s'},\bar{t'}))*(x_n,y_n) & (u,v) \\ =(x_n,y_n)*(u,v) & \end{array}$$

Thus,

$$((s,t),(\bar{s},\bar{t}))\otimes(u,v)=((s',t'),(\bar{s'},\bar{t'}))\otimes(u,v)\in D(S\times T)\otimes(S\times T)(u,v),$$

well done.

Case 2: If $n>m$, then

$$\begin{array}{ll} & (p_1,q_1)*(u,v)=(u,v) \\ ((s,t),(\bar{s},\bar{t}))*(p_1,q_1)=((s_1,t_1),(\bar{s}_1,\bar{t}_1))*(x_1,y_1) & (p_2,q_2)*(u,v)=(x_1,y_1)*(u,v) \\ ((s_1,t_1),(\bar{s}_1,\bar{t}_1))*(p_2,q_2)=((s_2,t_2),(\bar{s}_2,\bar{t}_2))*(x_2,y_2) & (p_3,q_3)*(u,v)=(x_2,y_2)*(u,v) \\ \vdots & \vdots \\ ((s_{m-1},t_{m-1}),(\bar{s}_{m-1},\bar{t}_{m-1}))*(p_m,q_m)=((s_m,t'),(\bar{s}_m,\bar{t'}))*(x_m,y_m) & (p_{m+1},1)*(u,v) \\ =(x_m,y_m)*(u,v) & \\ ((s_m,t'),(\bar{s}_m,\bar{t'}))*(p_m,q_m)=((s_{m+1},t'),(\bar{s}_{m+1},\bar{t'}))*(x_{m+1},1) & (p_{m+2},1)*(u,v) \\ =(x_{m+1},1)*(u,v) & \\ \vdots & \vdots \end{array}$$

$\big((s_{n-1},t'),(\bar{s}_{n-1},\bar{t'})\big) * (p_n,1) = \big((s',t'),(\bar{s'},\bar{t'})\big) * (x_n,1) \qquad (u,v) = (x_n,1) * (u,v)$

Thus, $\big((s,t),(\bar{s},\bar{t})\big) \otimes (u,v) = \big((s',t'),(\bar{s'},\bar{t'})\big) \otimes (u,v) \in D(S \times T) \otimes (S \times T)(u,v)$.

Case 3: If $n < m$, the same as the Case 2.

Therefore, the diagonal $(S \times T)$-act $D(S \times T)$ is principally weakly flat.

PROPOSITION 5.2 Let $(S,\cdot)$ and $(T,\circ)$ be two semigroups. Then the diagonal $(S \times T)$-act $D(S \times T)$ satisfies Condition (E) if and only if both the diagonal S-act $D(S)$ and the diagonal T-act $D(T)$ satisfy Condition (E).

PROOF We prove that the diagonal S-act $D(S)$ satisfies Condition (E). Suppose that $(s_1,s_2) \in D(S)$, $s,s' \in S$, and for any $(t_1,t_1) \in D(T)$, such that $(s_1,s_2) \cdot s = (s_1,s_2) \cdot s'$. Then have

$$((s_1,t_1),(s_2,t_1)) * (s,t) = ((s_1,t_1),(s_2,t_1)) * (s',t).$$

By assumption, there exist $((s_1',t_1'),(s_2',t_2')) \in D(S \times T)$ and $(x,y) \in S \times T$ such that

$$((s_1,t_1),(s_2,t_1)) = ((s_1',t_1'),(s_2',t_2')) * (x,y),\ (x,y) * (s,t) = (x,y) * (s',t).$$

Thus, we get

$$(s_1,s_2) = (s_1',s_2') \cdot x,\ x \cdot s = x \cdot s'.$$

Therefore, the diagonal S-act $D(S)$ satisfies Condition (E).

Similarly, suppose that for any $(t_1,t_2) \in D(T)$, $t,t' \in T$ and $(s_1,s_1) \in D(S)$, such that $(t_1,t_2) \circ t = (t_1,t_2) \circ t'$, it follows that

$$((s_1,t_1),(s_1,t_2)) * (s,t) = ((s_1,t_1),(s_1,t_2)) * (s,t').$$

So, the diagonal T-act $D(T)$ also satisfies Condition (E).

Conversely, assume that $s,s_1,p,p_1 \in S$, $t,t_1,q,q_1 \in T$, $((s,t),(s_1,t_1)) \in D(S \times T)$, $(p,q),(p_1,q_1) \in S \times T$, such that

$$((s,t),(s_1,t_1)) * (p,q) = ((s,t),(s_1,t_1)) * (p_1,q_1).$$

Then, we have

$$(s,t) * (p,q) = (s,t) * (p_1,q_1),\ (s_1,t_1) * (p,q) = (s_1,t_1) * (p_1,q_1),$$

So, $s \cdot p = s \cdot p_1$, $t \circ q = t \circ q_1$, $s_1 \cdot p = s_1 \cdot p_1$, $t_1 \circ q = t_1 \circ q_1$. Since the diagonal S-act $D(S)$ and the diagonal T-act $D(T)$ satisfy Condition (E), there exist $(s',s_1') \in D(S)$, $(t',t_1') \in D(T)$, $u \in S$, $v \in T$,~such that

$$(s,s_1) = (s',s_1') \cdot u,\ (t,t_1) = (t',t_1') \circ v,$$

and

$$u \cdot p = u \cdot p_1,\ v \circ q = v \circ q_1.$$

Thus, we get

$$((s,t),(s_1,t_1)) = ((s',t'),(s_1',t_1')) * (u,v),$$

and

$$(u,v)*(p,q)=(u,v)*(p_1,q_1),$$

where

$$((s',t'),(s_1',t_1'))\in D(S\times T),\ (u,v)\in S\times T.$$

PROPOSITION 5.3 Let $(S,\cdot)$ and $(T,\circ)$ be two semigroups. Then the diagonal $(S\times T)$-act $D(S\times T)$ satisfies Condition (P) if and only if both the diagonal S-act $D(S)$ and the diagonal T-act $D(T)$ satisfy Condition (P).

PROOF We prove that the diagonal S-act $D(S)$satisfies Condition (P). Suppose that for any $(s_1,s_2),(s_1',s_2')\in D(S),\ s,s'\in S,\ (t_1,t_1)\in D(T)$, such that $(s_1,s_2)\cdot s=(s_1',s_2')\cdot s'$. Then

$$((s_1,t_1),(s_2,t_1))*(s,t)=((s_1',t_1),(s_2',t_1))*(s',t).$$

By assumption, there exist $((s_1'',t_1''),(s_2'',t_2''))\in D(S\times T),\ (x,y),(x',y')\in S\times T$ such that

$$((s_1,t_1),(s_2,t_1))=((s_1'',t_1''),(s_2'',t_2''))*(x,y)$$
$$((s_1',t_1),(s_2',t_1))=((s_1'',t_1''),(s_2'',t_2''))*(x',y')$$

and

$$(x,y)*(s,t)=(x',y')*(s',t).$$

Thus, we get

$$(s_1,s_2)=(s_1'',s_2'')\cdot x,\ (s_1',s_2')=(s_1'',s_2'')\cdot x',\ x\cdot s=x'\cdot s'.$$

So, the diagonal S-act $D(S)$satisfies Condition (P).

Similarly, assume that $(t_1,t_2),(t_1',t_2')\in D(T),\ t,t'\in T$ and $(s_1,s_1)\in D(S),\ (t_1,t_2)\circ t=(t_1',t_2')\circ t'$, such that

$$((s_1,t_1),(s_1,t_2))*(s,t)=((s_1,t_1'),(s_1,t_2'))*(s,t').$$

Therefore, the diagonal T-act $D(T)$ satisfies Condition (P).

Conversely, for $s,s_1,s',s_1',p,p_1\in S,\ t,t_1,t',t_1',q,q_1\in T,\ ((s,t),(s_1,t_1)),\ ((s',t'),(s_1',t_1'))\in D(S\times T),(p,q),(p_1,q_1)\in S\times T$, such that

$$((s,t),(s_1,t_1))*(p,q)=((s',t'),(s_1',t_1'))*(p_1,q_1).$$

Then

$$(s,t)*(p,q)=(s',t')*(p_1,q_1),$$
$$(s_1,t_1)*(p,q)=(s_1',t_1')*(p_1,q_1),$$

So, we get $s\cdot p=s'\cdot p_1,\ t\circ q=t'\circ q_1,\ s_1\cdot p=s_1'\cdot p_1$ and $t_1\circ q=t_1'\circ q_1$. Thus,

$$(s,s_1)\cdot p=(s',s_1')\cdot p_1,\ \ (t,t_1)\circ q=(t',t_1')\circ q_1.$$

Since the diagonal S-act $D(S)$ and the diagonal T-act $D(T)$ satisfy Condition (P), there exist $(s'',s_1'')\in D(S),\ (t'',t_1'')\in D(T),\ u,u'\in S,\ v,v'\in T$ such that

$$(s,s_1)=(s'',s_1'')\cdot u,\ \ (s',s_1')=(s'',s_1'')\cdot u',$$

$$(t,t_1)=(t'',t_1'')\circ v,\ \ (t',t_1')=(t'',t_1'')\circ v',$$

and

$$u\cdot p=u'\cdot p_1,\ \ v\circ q=v'\circ q_1,$$

it follows from that

$$((s,t),(s_1,t_1))=((s'',t''),(s_1'',t_1''))*(u,v),$$
$$((s',t'),(s_1',t_1'))=((s'',t''),(s_1'',t_1''))*(u',v'),$$

and

$$(u,v)*(p,q)=(u',v')*(p_1,q_1).$$

Recall that in [17], let S and T be two semigroups, $\alpha: S\to End(T)$be a homomorphism from S to the semigroup of endomorphisms acting on T. For each $s\in S,\ t\in T$, we will denote by t^s the element of T. The semi-direct product $S\rtimes_\alpha T$ of S and T is the set $S\times T$ with multiplication of pairs defined by the rule

$$(s_1,t_1)(s_2,t_2)=(s_1s_2,t_1^{s_2}t_2).$$

THEOREM 5.4 Let S and T be two semigroups. If the diagonal $(S\rtimes_\alpha T)$-act $D(S\rtimes_\alpha T)$ is principally weakly flat, then the diagonal S-act $D(S)$ and the diagonal $T^e-actD(T^e)$ are principally weakly flat, where $T^e=\{t^e|t\in T,e\in E(S)\}$.

PROOF Suppose that the diagonal $(S\rtimes_\alpha T)$-act $D(S\rtimes_\alpha T)$ is principally weakly flat. We firstly verify the diagonal S-act $D(S)$ is principally weakly flat. Let $(s,\bar{s})u=(s',\bar{s'})u\in D(S), u\in S\ and\ t,v\in T$. Then we have

$$\big((s,t),(\bar{s},t)\big)(u,v)=\big((s',t),(\bar{s'},t)\big)(u,v)\in D(S\rtimes_\alpha T).$$

According to [14, Lemma 2.1] and principally weakly flatness, there exist $\big((s_i,t_i),(\bar{s}_i,\bar{t}_i)\big)\in D(S\rtimes_\alpha T),(p_i,q_i),(x_i,y_i)\in S\rtimes_\alpha T, i=1,2,\cdots,n$ such that

$$\begin{array}{ll} & (p_1,q_1)(u,v)=(u,v)\\ \big((s,t),(\bar{s},t)\big)(p_1,q_1)=\big((s_1,t_1),(\bar{s}_1,\bar{t}_1)\big)(x_1,y_1) & (p_2,q_2)(u,v)=(x_1,y_1)(u,v)\\ \big((s_1,t_1),(\bar{s}_1,\bar{t}_1)\big)(p_2,q_2)=\big((s_2,t_2),(\bar{s}_2,\bar{t}_2)\big)(x_2,y_2) & (p_3,q_3)(u,v)=(x_2,y_2)(u,v)\\ \vdots & \vdots\\ \big((s_{n-1},t_{n-1}),(\bar{s}_{n-1},\bar{t}_{n-1})\big)(p_n,q_n)=\big((s',t),(\bar{s'},t)\big)(x_n,y_n) & (u,v)=(x_n,y_n)(u,v)\end{array}$$

We obtain

$$\begin{array}{ll} & p_1u=u\\ (s,\bar{s})p_1=(s_1,\bar{s}_1)x_1 & p_2u=x_1u\end{array}$$

$$(s_1,\bar{s}_1)p_2 = (s_2,\bar{s}_2)x_2 \qquad p_3u = x_2u$$
$$\vdots \qquad\qquad \vdots$$
$$(s_{n-1},\bar{s}_{n-1})p_n = (s',\bar{s'})x_n \qquad u = x_nu.$$

So we have $(s,\bar{s}) \otimes u = (s',\bar{s'}) \otimes u \in D(S) \otimes Su$, Thus the diagonal S-act $D(S)$ is principally weakly flat.

On the other hand, let $(t^e,\bar{t}^e)v = (z^e,\bar{z}^e)v \in D(T^e), v \in T$ and $s,e \in S$. Then we have

$$\big((s,t),(s,\bar{t})\big)(e,v) = \big((s,z),(s,\bar{z})\big)(e,v) \in D(S \rtimes_\alpha T).$$

There exist $((s_i,t_i),(\overline{s_i},\overline{t_i})) \in D(S \rtimes_\alpha T), p_2,\cdots,p_n,x_1,\cdots,x_{n-1} \in S, q_1^e,\cdots,q_n^e,y_1^e,\cdots,y_n^e \in T^e$ such that

$$(e,q_1)(e,v) = (e,v)$$
$$((s,t),(s,\bar{t}))(e,q_1^e) = ((s_1,t_1),(\bar{s}_1,\bar{t}_1))(x_1,y_1^e) \qquad (p_2,q_2)(e,v) = (x_1,y_1)(e,v)$$
$$((s_1,t_1),(\bar{s}_1,\bar{t}_1))(p_2,q_2^e) = ((s_2,t_2),(\bar{s}_2,\bar{t}_2))(x_2,y_2^e) \qquad (p_3,q_3)(e,v) = (x_2,y_2)(e,v)$$
$$\vdots \qquad\qquad \vdots$$
$$((s_{n-1},t_{n-1}),(\bar{s}_{n-1},\bar{t}_{n-1}))(p_n,q_n^e) = ((s,z),(s,\bar{z}))(e,y_n^e) \qquad (e,v) = (e,y_n)(e,v)$$

Then, it follows that

$$q_1^e v = v$$
$$(t,\bar{t}^e)q_1^e = (t_1,\bar{t}_1)y_1^e \qquad q_2^e v = y_1^e v$$
$$(t_1,\overline{t_1})q_2^e = (t_2,\overline{t_2})y_2^e \qquad q_3^e v = y_2^e v$$
$$\vdots \qquad\qquad \vdots$$
$$(t_{n-1},\overline{t_{n-1}})q_n^e = (z^e,\bar{z}^e)y_n^e \qquad v = y_n^e v.$$

So $(t^e,\bar{t}^e) \otimes v = (z^e,\bar{z}^e) \otimes v \in D(T^e) \otimes T^e v$. Thus, the diagonal $T^e - act\ D(T^e)$ is principally weakly flat.

The wreath product of semigroups is also used as a diagonal act, but it is still unclear whether the diagonal acts on the wreath product of semigroups also has the above properties, which has become the next aspect that we need to study.

References:

[1] Anderson, F.W., Fuller, K.R.(1974). Rings and categories of modules. In: Graduate Texts in Mathematics, vol.13. Springer, New York.

[2] Bulman-Fleming, S., Gilmour, A. (2009). Flatness properties of diagonal acts over monoids. Semigroup Forum. 79, 298-314.

[3] Bulman-Fleming, S., McDowell, K. (1989). Problem E3311. Amer. Math. Monthly. 96, 155.

[4] Bulman-Fleming, S., McDowell, K.(1990). Solution. Amer. Math. Monthly. 97, 617.

[5] Ebrahimi, M. M., Mahmoudi, M.(1995). When is the category of separated M -sets a quasitopos or a topos? Bull. Iranian Math. Soc. 21, 25-33.

[6] Gallagher, P. (2006). On the finite and nonfinite generation of diagonal acts. Comm. Algebra. 34, 3123-3137.

[7] Gallagher, P., Ruskuc, N.(2005). Generation of diagonal acts of some semigroups of transformations and relations. Bull. Aust. Math. Soc. 72, 139-146.

[8] Golchin, A., Abbasi, M., Mohammadzadeh, H. (2021). On a generalization of weak flatness. Asian-Eur. J. Math. 14, 1-20.

[9] Howie, M.(1995). Fundamentals of Semigroup Theory. Oxford Science Publications: Oxford.

[10] Kilp, M., Knauer, U., Mikhalev, A.(2000). Monoids, Acts and Categories: with Applications to Wreath Products and Graphs. New York: Walter de Gruyter..

[11] Laan, V., Marki, L.(2012). Morita invariants for semigroups with local units. Monatsh Math. 166, 441-451 .

[12] Laan, V., Reimaa, U., Tart, L., Teor, E. (2021). Flatness properties of acts over semigroups. Categories and General Algebraic Structures with Applications. 15, 59-92.

[13] Lawson, M.V.(2011). Morita equivalence of semigroups with local units. J. Pure Appl. Algebra 215, 455 – 470 .

[14] Liang, X. L., Khosravi, R., Zhao, X. Z. (2021). On the homological classification of semigroups with local units. Bull. Malays. Math. Sci. Soc. 44, 2893-2917.

[15] Laan, V., Marki, L., Reimaa, u.(2018). Morita equivalence of semigroups revisited: Firm semigroups. J. Algebra 505, 247-270 .

[16] Nagy, A., Toth, C.(2024). On Rees matrix semigroups $M(S;I,\Lambda;P)$ over semigroups S in which I is a singleton. Comm. Algebra. 52, 4432-4443 .

[17] Nico, W R.(1983). On the regularity of semidirect products. J. Algebra. 80, 29-36 .

[18] Robertson, E. F., Ruskuc, N., Thomson, M.R.(2001). On diagonal acts of monoids. Bull. Aust. Math. Soc. 63, 167-175 .

[19] Talwar, s.(1995). Morita equivalence for semigroups. J. Ausral. Math. Soc. (Ser. A) 59, 81-111.

[20] Talwar, s.(1996). Strong Morita equivalence and a generalisation of the Rees theorems. J. Algebra 181, 371-394.